Introduction to Programming (in C++)

Algorithms on sequences. Reasoning about loops: Invariants.

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Outline

- Algorithms on sequences
  - Treat-all algorithms
  - Search algorithms

- Reasoning about loops: invariants

Maximum of a sequence

- Write a program that tells the largest number in a non-empty sequence of integers.

```
// Pre:  a non-empty sequence of integers is
//       ready to be read at cin
// Post: the maximum number of the sequence has been
//       written at the output
```

`Assume the input sequence is: 23 12 -16 34 25`

elem:	-	12	-16	34	25
m:	23	23	23	34	34

```
// Invariant: m is the largest number read
//            from the sequence
```

Maximum of a sequence

```
int main() {
    int m;

    int elem;
    cin >> m;
    // Inv: m is the largest element read
    //      from the sequence
    while (cin >> elem) {
        if (elem > m) m = elem;
    }
    cout << m << endl;
}
```

Why is this necessary?

Checks for end-of-sequence and reads a new element.

Reading with cin

- The statement `cin >> n` can also be treated as a Boolean expression:
  - It returns *true* if the operation was successful
  - It returns *false* if the operation failed:
    - no more data were available (EOF condition) or
    - the data were not formatted correctly (e.g. trying to read a double when the input is a string)

- The statement:

```
cin >> n
```

can be used to detect the end of the sequence and read a new element simultaneously. If the end of the sequence is detected, **n** is not modified.

Finding a number greater than n

- Write a program that detects whether a sequence of integers contains a number greater than n.

```
// Pre:  at the input there is a non-empty sequence of
//       integers in which the first number is n.
// Post: writes a Boolean value that indicates whether
//       a number larger than n exists in the sequence.
```

Assume the input sequence is: 23 12 -16 24 25

num:	-	12	-16	24
n:	23	23	23	23
found:	false	false	false	true

```
// Invariant: "found" indicates that a value greater than
//            n has been found.
```

Finding a number greater than n

```
int main() {
    int n, num;
    cin >> n;
    bool found = false;

    // Inv: found indicates that a number
    //      greater than N has been found
    while (not found and cin >> num) {
        found = num > n;
    }
    cout << found << endl;
}
```

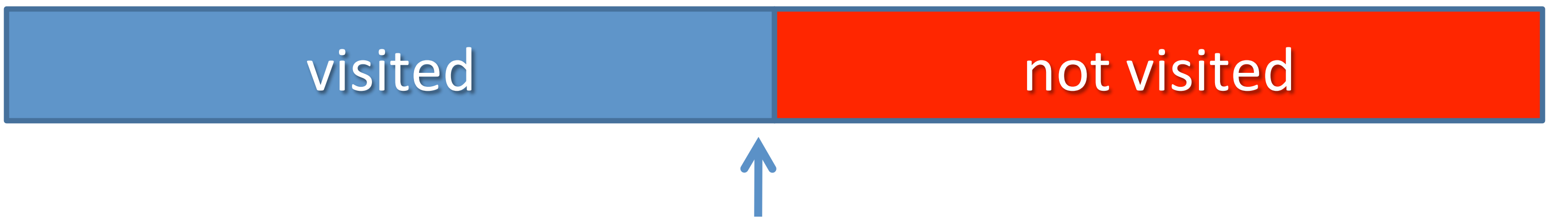
Algorithmic schemes on sequences

- The previous examples perform two different operations on a sequence of integers:
  - Finding the maximum number
  - Finding whether there is a number greater than N

- They have a distinctive property:
  - The former requires all elements to be visited
  - The latter requires one element to be found

Treat-all algorithms

- A classical scheme for algorithms that need to treat all the elements in a sequence




```
Initialize (the sequence and the treatment)
// Inv: The visited elements have been treated
while (not end of sequence) {
    Get a new element e;
    Treat e;
}
```

Search algorithms

- A classical scheme for algorithms that need to find an element with a certain property in a sequence

```
bool found = false;
Initialize;
// Inv: “found” indicates whether the element has been
//      found in the visited part of the sequence
while (not found and not end of sequence) {
    Get a new element e;
    if (Property(e)) found = true;
}
// “found” indicates whether the element has been found.
// “e” contains the element.
```

Longest repeated subsequence

- Assume we have a sequence of strings

  cat dog bird cat bird bird cat cat cat dog mouse horse

- We want to calculate the length of the longest sequence of repetitions of the first string. Formally, if we have a sequence of strings

$$s_1, s_2, \ldots, s_n$$

we want to calculate

$$\max\{j - i + 1 : 1 \leq i \leq j \leq n \ \wedge \ s_i = s_{i+1} = \cdots = s_{j-1} = s_j = s_1\}.$$

Longest repeated subsequence

```
// Specification: see previous slide
// Variable to store the first string.
string first;
cin >> first;
string next; // Visited string in the sequence
// Length of the current and longest subsequences
int length = 1;
int longest = 1;
// Inv: “length” is the length of the current subsequence.
//      “longest” is the length of the longest subsequence
//      visited so far.
while (cin >> next) {
    if (first != next) length = 0; // New subsequence
    else {
        // The current one is longer
        length = length + 1;
        if (length > longest) longest = length;
    }
}
// “longest” has the length of the longest subsequence
```

Search in the dictionary

- Assume we have a sequence of strings representing words. The first string is a word that we want to find in the dictionary that is represented by the rest of the strings. The dictionary is ordered alphabetically.

- Examples:

  dog ant bird cat cow dog eagle fox lion mouse pig rabbit shark whale yak

  frog ant bird cat cow dog eagle fox lion mouse pig rabbit shark whale yak

- We want to write a program that tells us whether the first word is in the dictionary or not.

Search in the dictionary

```
// Specification: see previous slide
// First word in the sequence (to be sought).
string word;
cin >> word;

// A variable to detect the end of the search
// (when a word is found that is not smaller than “word”).
bool found = false;

// Visited word in the dictionary (initialized as empty for
// the case in which the dictionary might be empty).
string next = “”;

// Inv: not found => the visited words are smaller than “word”
while (not found and cin >> next) found = next >= word;
// “found” has detected that there is no need to read the rest of
// the dictionary
found = word == next;
// “found” indicates that the word was found.
```

Increasing number

- We have a natural number *n*. We want to know whether its representation in base 10 is a sequence of increasing digits.

- Examples:

134679	→ increasing
56688	→ increasing
3	→ increasing
134729	→ non-increasing

Increasing number

```
// Pre: n >= 0
// Post: It writes YES if the sequence of digits representing n (in base 10)
// is increasing, and it writes NO otherwise

int main() {
    int n;
    cin >> n;
    // The algorithm visits the digits from LSB to MSB.
    bool incr = true;
    int previous = 9;  // Stores a previous “fake” digit

    // Inv: n contains the digits no yet treated, previous contains the
    //      last treated digit (that can never be greater than 9),
    //      incr implies all the treated digits form an increasing sequence
    while (incr and n > 0) {
        int next = n%10;
        incr = next <= previous;
        previous = next;
        n /= 10;
    }

    if (incr) cout << “YES” << endl;
    else cout << “NO” << endl;
}
```

Insert a number in an ordered sequence

- Read a sequence of integers that are all in ascending order, except the first one. Write the same sequence with the first element in its correct position.

- Note: the sequence has at least one number. The output sequence must have a space between each pair of numbers, but not before the first one or after the last one.

- Example

  Input: **15** 2 6 9 12 18 20 35 75
  Output: 2 6 9 12 **15** 18 20 35 75

- The program can be designed with a combination of search and treat-all algorithms.

Insert a number in an ordered sequence

```
int first;
cin >> first;

bool found = false;       // controls the search of the location
int next;                 // the next element in the sequence

// Inv: All the read elements that are smaller than the first have been written
//      not found => no number greater than or equal to the first has been
//      found yet
while (not found and cin >> next) {
    if (next >= first) found = true;
    else cout << next << " ";
}

cout << first;

if (found) {
    cout << " " << next;
    // Inv: all the previous numbers have been written
    while (cin >> next) cout << " " << next;
}
cout << endl;
```

REASONING ABOUT LOOPS: INVARIANTS

Invariants

- Invariants help to …
  - Define how variables must be initialized before a loop
  - Define the necessary condition to reach the post-condition
  - Define the body of the loop
  - Detect whether a loop terminates

- It is crucial, but not always easy, to choose a good invariant.

- Recommendation:
  - Use invariant-based reasoning for all loops (possibly in an informal way)
  - Use formal invariant-based reasoning for non-trivial loops

General reasoning for loops

```
Initialization;
// Invariant: a proposition that holds
//     * at the beginning of the loop
//     * at the beginning of each iteration
//     * at the end of the loop
// Invariant
while (condition) {
    // Invariant ∧ condition
    Body of the loop;
    // Invariant
}
// Invariant ∧  ¬ condition
```

Example with invariants

- Given $n \geq 0$, calculate $n!$

- Definition of factorial:

$$n! = 1 * 2 * 3 * \ldots * (n-1) * n$$

(particular case: 0! = 1)

- Let's pick an invariant:
  - At each iteration we will calculate f = i!
  - We also know that $i \leq n$ at all iterations

Calculating n!

```
// Pre: n ≥ 0
// It writes n!
int main() {
    int n;
    cin >> n;
    int i = 0;
    int f = 1;
    // Invariant: f = i! and i ≤ n
    while (i < n) {
        // f = i! and i < n
        i = i + 1;
        f = f*i;
        // f = i! and i ≤ n
    }
    // f = i! and i ≤ n and i ≥ n
    // f = n!
    cout << f << endl;
}
```

Reversing digits

- Write a program that reverses the digits of a number (representation in base 10)

- Examples:

```
35276  →  67253
   19  →  91
    3  →  3
    0  →  0
```

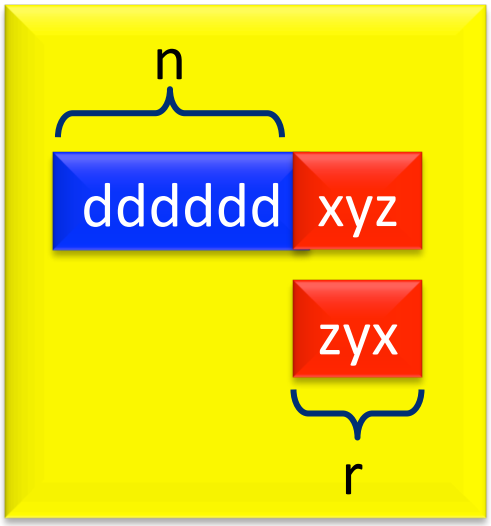
Reversing digits

```
// Pre: n ≥ 0
// Post: It writes n with reversed digits (base 10)

int main() {
    int n;
    cin >> n;
    int r;

    r = 0;
    // Invariant (graphical): →
    while (n > 0) {
        r = 10*r + n%10;
        n = n/10;
    }

    cout << r << endl;
}
```



Calculating π

- π can be calculated using the following series:

$$\frac{\pi}{2} = 1 + \frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 5} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} + \cdots$$

- Since an infinite sum cannot be computed, it may often be sufficient to compute the sum with a finite number of terms.

Calculating π

```
// Pre: nterms > 0
// It writes an estimation of π using nterms terms
// of the series

int main() {
    int nterms;
    cin >> nterms;
    double sum = 1;            // Approximation of π/2
    double term = 1;           // Current term of the sum

    // Inv: sum is an approximation of π/2 with k terms,
    //      term is the k-th term of the series.
    for (int k = 1; k < nterms; ++k) {
            term = term*k/(2.0*k + 1.0);
        sum = sum + term;
    }
    cout << 2*sum << endl;
}
```

Calculating π

- π = 3.14159265358979323846264338327950288...
- The series approximation:

nterms	Pi(nterms)
1	2.000000
5	3.098413
10	3.140578
15	3.141566
20	3.141592
25	3.141593